Taking Schrödinger Seriously
Module 1

What Is the Schrödinger Equation?

Maxime Desalle

CONJECTURE UNIVERSITY

What Is the Schrödinger Equation?

Below is the Schrödinger equation. If you have never seen it before, take a moment to look at it. If you have seen it before, look at it again. This equation sets the foundations of quantum mechanics: it governs atoms, molecules, chemistry, and everything they make up.

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi}$$

At first glance, the equation seems disappointingly simple. For a theory that describes all of matter, where is the complexity?

The complexity is hidden in the symbols, each compressing enormous meaning: the wave function Ψ, the Hamiltonian $\hat{H}$, the imaginary unit i, Planck's constant $\hbar$, and the symbol for how fast Ψ is changing (written $\frac{\partial\Psi}{\partial t}$).

By the end of this module, you will be able to:

- Identify each symbol in the Schrödinger equation and explain what it represents,
- Describe what the wave function is and why it must be complex (here, *complex* means ‘having both real and imaginary components’),
- Explain why the equation uses only the first derivative of Ψ, and what this implies about determinism,
- Read the Hamiltonian as a specification of what physical situation the system is in, and
- Articulate what the equation says, and what it does *not* say.

In the most basic terms, the Schrödinger equation tells us how quantum states change over time. Give it a state now, and it tells you the state at any future moment: it is a law of motion for reality.

There is neither randomness nor collapse in this equation. Rather, this equation sets out the smooth, continuous, and deterministic evolution of the wave function.

The Wave Function Ψ

The symbol Ψ is the Greek letter psi (pronounced “sigh”) and denotes the wave function, the central object in quantum mechanics. The wave function is what describes physical reality in quantum mechanics. There is a wave function for a single particle, for an atom, for a molecule, and ultimately for the entire universe.

The wave function is a mathematical function: it takes an input and produces an output. For example, in the case of a single particle, the input would be a position in space (and time), and the output would be a number (specifically, a complex number, which we'll explain shortly).

Essentially, you are asking the function, "What is the wave function at this point?" and it returns that value. The function is called a "wave" function because it behaves like a wave: it oscillates, has peaks and troughs, and when multiple wave functions overlap, they interfere with each other.

The wave function is complex in the mathematical sense. To understand what this means, think of an arrow.

At any point, the wave function is like an arrow with two properties: its magnitude (how long it is) and its phase (in which direction it points). The magnitude tells you how "big" the wave function is at that point. The phase tells you where it is in a cycle of oscillation.

These arrows live on a two-dimensional plane called the complex plane. The horizontal axis represents ordinary "real" numbers (like 3 or -7.5). The vertical axis represents "imaginary" numbers. Any point on this plane, any arrow, can be described by how far right it goes and how far up it goes.

This is what a complex number is: a way of writing where the arrow points. If the arrow goes 3 units right and 2 units up, we write $3 + 2i$. The i means "in the vertical direction." Mathematically, i is defined as $\sqrt{-1}$, but for our purposes, just think of it as "pointing up" on the complex plane.

The complexity of the wave function is essential. Without complex numbers, there would be no interference, and quantum mechanics would not work. We'll see exactly why when we discuss the imaginary unit i in detail.

So, what does a wave function actually look like? Let's look at an example.
A particle confined to a box of length L, at position x, can have the following wave function, for example:

$$\Psi(x) = \sqrt{\frac{2}{L}} sin\left(\frac{\pi x}{L}\right)$$

This describes the particle inside the box. Outside the box, the wave function is zero: the particle is trapped, with no chance of being found beyond the walls.

Don't let this intimidate you. Let's break it down.

The sine function, $sin\left(\frac{\pi x}{L}\right)$, determines the shape. It starts at zero at one wall, rises to a peak in the middle, and falls back to zero at the other wall. The particle is most likely to be found near the center.

The factor $\sqrt{\frac{2}{L}}$ handles normalization: it ensures the total probability of finding the particle somewhere in the box equals exactly 1, or 100%. The particle must be somewhere, and this factor guarantees that the math respects that.

How do we get from the wave function to probability? We take the magnitude of Ψ and square it, written $|\Psi|^2$. This is called the probability density. Where $|\Psi|^2$ is large, the particle is more likely to be found. Where it's small, the particle is unlikely to be found.

The word "density" matters: $|\Psi|^2$ tells you how probability is concentrated at each point. To find the probability of the particle being in some region, you add up the density across that region. And when you add up the density across the entire box, you always get 1.

The wave function exists in an abstract mathematical space called *Hilbert space*, which we will discuss in Module 2.

The Time Derivative $\frac{\partial \Psi}{\partial t}$

How fast is the wave function changing? That's what the left side of the Schrödinger equation tells us. The expression $\frac{\partial \Psi}{\partial t}$ is called the time derivative of the wave function, and it represents the rate of change at any moment.

If you haven't encountered derivatives before, think of them as measuring how fast something is changing. The derivative of your position with respect to time is your velocity: how fast you're moving. The derivative of your bank balance with respect to time is how fast you're gaining or losing money. The derivative of the wave function with respect to time, $\frac{\partial \Psi}{\partial t}$, is how fast the wave function is changing.

The Schrödinger equation is first-order in time: it involves only this first derivative (how fast Ψ is changing), not the second derivative (how fast that rate itself is changing). Why does this matter? In classical mechanics, predicting a ball's trajectory requires three pieces of information: position, velocity, and the forces acting on it. The Schrödinger equation requires only two: the wave function in its current state and the Hamiltonian, which specifies the physical setup (more about the Hamiltonian below). The equation then determines Ψ at any future moment.

This is determinism in its purest form: the present state, evolving under a known Hamiltonian, fully determines all future states.

The Imaginary Unit *i*

We've already met i, the imaginary unit, when discussing complex numbers. Now let's see why it appears in the Schrödinger equation.

Think about what kinds of behavior a physics equation can produce. Simple equations describing change over time give either growth or decay. A bank account with compound interest grows exponentially. A radioactive substance decays exponentially. Ordinary equations with real numbers produce solutions that explode or shrink.

The wave function does neither. It oscillates.

When i appears in an equation, the solution doesn't grow or decay—it rotates. In the complex plane, the wave function traces circles, cycling endlessly without ever changing in magnitude. And as the arrow rotates, its components rise and fall. That rising and falling is oscillation, and oscillation is what makes a wave.

The Schrödinger equation dictates that, as time passes, the phase changes but the magnitude does not. The arrow rotates without growing or shrinking. This oscillation makes interference possible. When two wave functions meet, their phases determine whether they reinforce or cancel. If the phases are aligned, the waves add up. If the phases oppose, the waves cancel out.

Without oscillation, there would be no phase. Without phase, there would be no interference. And without interference, there would be no quantum mechanics.

Let's work through a concrete example.

Suppose two wave functions have the same magnitude of 1 but different phases.

If both have phase 0° (arrows pointing the same direction), we write $\Psi_1 = 1$ and $\Psi_2 = 1$. Combined:

$$\Psi_1 + \Psi_2 = 1 + 1 = 2$$

This is constructive interference: the waves reinforce each other.

Now suppose one has phase 180° (arrow pointing backward). We write $\Psi_1 = 1$ and $\Psi_2 =- 1$. Combined:

$$\Psi_1 + \Psi_2 = 1 + (- 1) = 0$$

This is destructive interference: the waves cancel completely.

What about phases in between? Suppose the phases are 90° apart. A phase of 90° means the arrow points straight up on the complex plane, which we write as i. So $\Psi_1 = 1$ and $\Psi_2 = i$. Combined:

$$\Psi_1 + \Psi_2 = 1 + i$$

To find the magnitude of a complex number $a + bi$, we compute $\sqrt{a^2 + b^2}$. Here, $a = 1$ and $b = 1$, so the magnitude is $\sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$.

This is between 0 (complete cancellation) and 2 (complete reinforcement). Intermediate phases give intermediate results.

Planck's Constant $\hbar$

Beside the i in the Schrödinger equation is the symbol $\hbar$, called "h-bar." This is the reduced Planck constant.

Planck's constant is the specific number: $\hbar \approx 1.055 \times 10^{-34} J \cdot s$.

The unit $J \cdot s$ stands for joule-seconds, or energy multiplied by time. Planck's constant is a fundamental constant of nature, just like the speed of light. It is not something that we derive but rather is a constant that we measure.

Technically, $\hbar = \frac{h}{2\pi}$, where h is the original Planck constant. The factor of 2π appears because a full rotation is 2π radians. Using $\hbar$ makes equations involving oscillations cleaner.

What does $\hbar$ do in the Schrödinger equation? It sets the rate at which quantum phase evolves relative to energy.

For everyday objects like a baseball or a car, the energies involved are enormous compared to $\hbar$, so the phase rotates trillions of trillions of times per second. These oscillations are far too rapid to observe, and the world appears classical. For an electron in an atom, the energies are much smaller, so the phase evolves slowly enough for quantum effects to be observed.

The Hamiltonian $\hat{H}$

On the right side of the Schrödinger equation sits the term $\hat{H}$, called the Hamiltonian.

In classical physics, the Hamiltonian denotes the total energy of a system: kinetic energy plus potential energy. A ball rolling on a hill has kinetic energy from its motion and potential energy from its height. Add them together, and you get the Hamiltonian.

In quantum mechanics, the Hamiltonian plays an analogous role, but there is one crucial difference: it is not a number but rather an *operator*.

An operator is a mathematical instruction that transforms one function into another. For example, the derivative operator $\frac{d}{dx}$ transforms the function x^2 into $2x$. You feed in one function, and out comes a different one.

The little hat on $\hat{H}$ reminds us this is an operator. When we write $\hat{H}\Psi$, we mean "apply the operator $\hat{H}$ to the wave function Ψ."

The standard Hamiltonian for a single particle is:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

The first term, $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$, is the kinetic energy operator. Why does $\frac{\partial^2}{\partial x^2}$ represent kinetic energy? The second derivative measures how "wiggly" a function is—how sharply it curves. A more wiggly wave function has a shorter wavelength, which means higher momentum, which means higher kinetic energy. The second term, $V(x)$, is the potential energy at position x.

Here are some examples of what $V(x)$ might look like for various systems:

- For a free particle, $V(x) = 0$, as there are no forces, just kinetic energy.
- For a particle in a box, $V(x) = 0$ inside the box, and $V(x) = \infty$ outside the box, as the particle is trapped when inside.
- For a harmonic oscillator, $V(x) = \frac{1}{2}kx^2$ (spring-like restoring force).

Writing down the Hamiltonian means specifying what forces or constraints are acting on the particle. The Schrödinger equation then tells you how the wave function evolves under those conditions.

The Meaning of It All

Now that we've examined every symbol, let's step back and review the equation as a whole:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

The left side of the equation describes how the wave function changes. The time derivative $\frac{\partial\Psi}{\partial t}$ gives the rate of change, i ensures oscillation rather than growth or decay, and $\hbar$ sets the scale of quantum effects.

The right side of the equation describes the causes of the wave function's changes. The Hamiltonian $\hat{H}$ encodes the physics: which forces act, which constraints exist, which interactions are present.

That's the entirety of the Schrödinger equation. Its state evolves according to the Hamiltonian and nothing more.

You may have noticed that there is no term for observation, no term for measurement, no term for collapse, no term for randomness. There's no special rule that's applied when a physicist looks at the system, no boundary between quantum and classical worlds. Instead, there is just the smooth, continuous, and deterministic evolution of the wave function, governed by the Hamiltonian, for all time.

Collapse interpretations (often called Copenhagen) say otherwise. According to these interpretations, Ψ encodes probabilities such that when a measurement occurs, the wave function *collapses* to match the observed result, and the other possibilities simply vanish.

But the mathematics of quantum mechanics does not require collapse. Collapse was added to explain why we observe only one outcome. Yet not observing other outcomes does not make them unreal. Schrödinger shared this perspective:

> "*To my mind it is patently absurd to let the wave function be controlled in two entirely different ways, at times by the wave equation, but occasionally by direct interference of the observer, not controlled by the wave equation.*"
>
> – Erwin Schrödinger, Dublin Seminar (1952)[1]

If collapse isn't real, where does the apparent randomness come from? Why do we observe just one outcome when measuring something? The answers involve entanglement, branching, and decoherence, but we will get to that in a future module.

For now: the equation says what it says, and that we must take it at face value.

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[1] Published in *The Interpretation of Quantum Mechanics*, ed. Michel Bitbol (Ox Bow Press, 1995), p. 35.

Conclusion

The Schrödinger equation is:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

The equation’s symbols each carry meaning:

- Ψ is the wave function, the complete physical state of a quantum system.
- $\frac{\partial \Psi}{\partial t}$ is the time derivative, describing how the wave function changes.
- $\hat{H}$ is the Hamiltonian, encoding the energy and interactions of the system.
- i is the imaginary unit, ensuring oscillation rather than growth or decay.
- $\hbar$ is Planck′s constant, setting the scale of quantum effects.

Schrödinger’s equation is deterministic, continuous, and universal; we must take it literally.

In upcoming modules, we will explore where the wave function lives, what the Hamiltonian contains, and how time evolution works. Later, we will confront what makes quantum mechanics seem strange: superposition, entanglement, measurement, and decoherence. If this sounds complicated, remember that it all follows from the equation we have just deconstructed.